

# AN ARRANGEMENT OF PSEUDOCIRCLES NOT REALIZABLE WITH CIRCLES

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ABSTRACT. We present an arrangement of five pseudocircles that cannot be realized with (proper) circles.

## 1. INTRODUCTION

By a *pseudocircle* we mean a simple closed Jordan curve in the plane.

**Definition 1.1.** An *arrangement of pseudocircles* is a finite set  $\Gamma = \{\gamma_1, \dots, \gamma_n\}$  of simple closed Jordan curves in the plane such that

- ▷ no three curves meet each other at the same point,
- ▷ if two pseudocircles  $\gamma_i, \gamma_j$  have a point  $P$  in common, they cross each other in that point, i. e. every neighborhood of  $P$  contains points of  $\gamma_i$  in the interior of  $\gamma_j$  as well as in the exterior of  $\gamma_j$ ,
- ▷ each pair of curves intersects at most 2 times.

An arrangement is said to be *complete* if each two pseudocircles intersect.

Given an arrangement of pseudocircles, we may consider the intersection points of the pseudocircles as vertices and the curves between the intersections as edges. Thus we obtain in a natural way an embedding of a graph and hence a cell complex. Two arrangements are said to be *isomorphic* if they have the same associated cell complex.

Obviously, arrangements of pseudocircles are a generalization of arrangements of (proper) circles. However, from the combinatorial point of view it is not clear whether the class of arrangements of pseudocircles is a proper extension of the class of arrangements of circles. Put in another way, the question is whether for every arrangement of pseudocircles there is an isomorphic arrangement of circles. There is an analogous problem concerning the stretchability of arrangements of pseudolines. That is, given an arrangement of pseudolines, is there a combinatorially equivalent arrangement of straight lines? In 1980, Goodman and Pollack [2] proved Grünbaum's conjecture that all arrangements of at most eight pseudolines are stretchable, so that some known non-stretchable arrangements of nine pseudolines are minimal in that sense (cf. [1], p.259ff). These arrangements also guarantee the existence of arrangements of (nine) pseudocircles on the sphere that cannot be realized as arrangements of great circles (cf. [1], p.249, 259ff). However, the problem of "straightening" arrangements of pseudocircles in the plane is a different matter.

In this paper, we settle the question by showing that there is an arrangement of five pseudocircles that is not isomorphic to any arrangement of circles. We conjecture that this example is minimal as well. In [3], we have shown that all five complete arrangements

of three and all 72 complete arrangements of four pseudocircles are realizable with proper circles. Thus, if there is a smaller example it is not complete.

## 2. THE ARRANGEMENT

Consider the arrangement of pseudocircles in Figure 1.

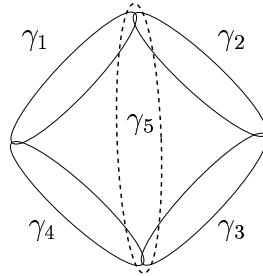


FIGURE 1. An arrangement of pseudocircles.

**Theorem 2.1.** *The arrangement in Figure 1 cannot be realized with circles.*

For the proof of Theorem 2.1 we shall need some simple observations expressed in the following two lemmata.

**Lemma 2.1.** *Let  $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$  be an arrangement of four circles with centers  $C_1, C_2, C_3, C_4$  that is isomorphic to the arrangement of the pseudocircles  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  in Figure 1. Let  $S_i$  and  $S'_i$  be the intersection points of  $\gamma_i$  and  $\gamma_{i+1}$  for each  $i \in \{1, 2, 3, 4\}$  modulo 4, such that  $S'_i$  lies on the boundary of the unbounded region (see Figure 2).*

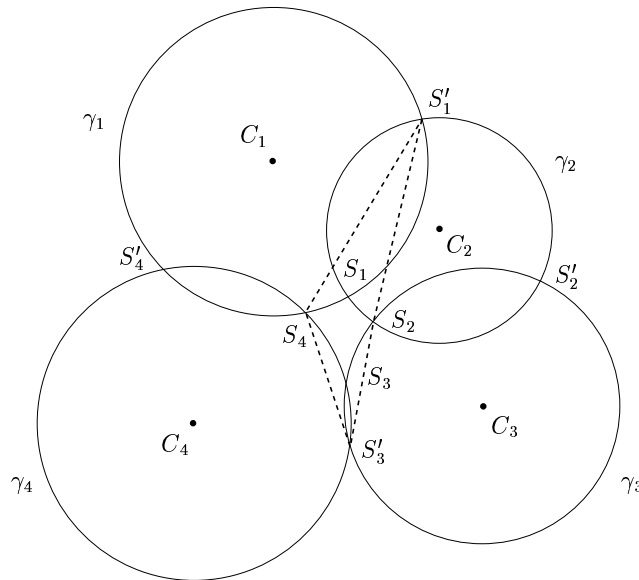


FIGURE 2. Illustration of Lemma 2.1.

Then in the quadrangle  $S'_1S_2S'_3S_4$  the sum of the angles at  $S_2$  and  $S_4$  is larger than the sum of the angles at  $S'_1$  and  $S'_3$ , i.e.  $\angle(S'_3S_4S'_1) + \angle(S'_1S_2S'_3) > \angle(S_4S'_1S_2) + \angle(S_2S'_3S_4)$ .<sup>1</sup>

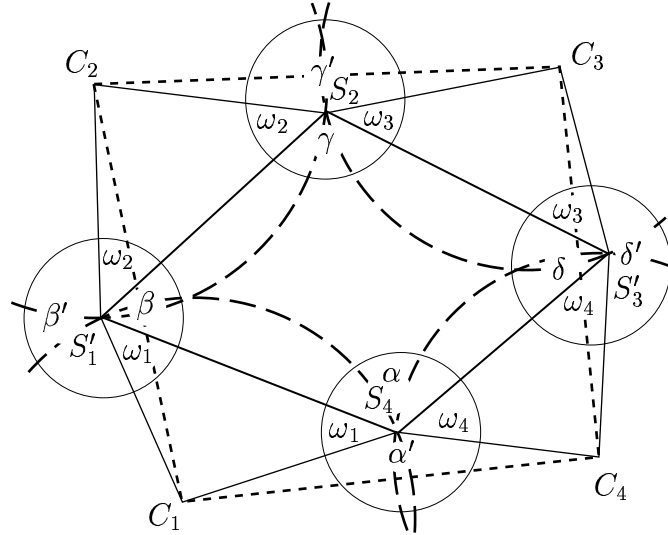


FIGURE 3. Illustration of proof of Lemma 2.1.

*Proof.* Consider the points  $S'_1, S_2, S'_3, S_4$  together with the centers  $C_1, C_2, C_3, C_4$ . Note that the triangles  $\Delta_1 = S_4C_1S'_1$ ,  $\Delta_2 = S'_1C_2S_2$ ,  $\Delta_3 = S_2C_3S'_3$  and  $\Delta_4 = S'_3C_4S_4$  are isosceles. Thus, let  $\omega_i$  be the two angles in  $\Delta_i$  lying opposite to  $C_i$ . Furthermore we set  $\alpha := \angle(S'_3S_4S'_1)$ ,  $\beta := \angle(S_4S'_1S_2)$ ,  $\gamma := \angle(S'_1S_2S'_3)$  and  $\delta := \angle(S_2S'_3S_4)$ . Finally, the exterior angles at the points  $S'_1, S_2, S'_3, S_4$  are denoted by  $\alpha', \beta', \gamma', \delta'$  (cf. Figure 3). Note that the points  $S_1, S_2, S_3, S_4$  are always contained in the interior of the quadrangle  $C_1C_2C_3C_4$ , while the points  $S'_1, S'_2, S'_3, S'_4$  lie outside. Hence, the angles  $\alpha', \gamma'$  are  $< 180^\circ$  while  $\beta', \delta' > 180^\circ$ , so that  $\alpha' + \gamma' < \beta' + \delta'$ . Thus,

$$\begin{aligned} (360^\circ - \alpha - \omega_1 - \omega_4) + (360^\circ - \gamma - \omega_2 - \omega_3) &= \alpha' + \gamma' < \\ < \beta' + \delta' = (360^\circ - \beta - \omega_1 - \omega_2) + (360^\circ - \delta - \omega_3 - \omega_4), \end{aligned}$$

whence  $\alpha + \gamma > \beta + \delta$ .  $\square$

**Lemma 2.2.** Let  $P_1, P_2, P_3, P_4, Q$  be five points in the plane such that the angles  $\angle(P_1QP_2)$ ,  $\angle(P_2QP_3)$ ,  $\angle(P_3QP_4)$ ,  $\angle(P_4QP_1)$  are all  $\leq 180^\circ$ . Then  $Q \in \text{conv}(P_1, P_2, P_3, P_4)$ .<sup>2</sup>

*Proof.* Let  $P_1, P_2, P_3, P_4, Q$  be five points in the plane such that  $Q$  is not contained in  $\text{conv}(P_1, P_2, P_3, P_4)$ . We show that one of the angles  $\angle(P_1QP_2)$ ,  $\angle(P_2QP_3)$ ,  $\angle(P_3QP_4)$ ,  $\angle(P_4QP_1)$  is  $> 180^\circ$ . Since  $Q \notin \text{conv}(P_1, P_2, P_3, P_4)$ , there is a straight line  $h$  that separates  $Q$  from  $\text{conv}(P_1, P_2, P_3, P_4)$ . Let  $P'_1, P'_2, P'_3, P'_4$  be the central projections of  $P_1, P_2, P_3, P_4$  from  $Q$  onto  $h$ . Now, if the counterclockwise order of the points  $P'_i$  on  $h$  relative to  $Q$  is  $P'_1, P'_2, P'_3, P'_4$ , then  $\angle(P_4QP_1) = \angle(P'_4QP'_1) > 180^\circ$ . Otherwise, there is

<sup>1</sup>In the following we consider all angles to be  $\in [0^\circ, 360^\circ)$  and oriented counterclockwise.

<sup>2</sup> $\text{conv } A$  denotes the convex hull of  $A$ .

an  $i \in \{1, 2, 3\}$  such that  $P'_{i+1}$  occurs before  $P'_i$  in the considered order. In this case,  $\angle(P_iQP_{i+1}) = \angle(P'_iQP'_{i+1}) > 180^\circ$ .  $\square$

Finally, we will also make use of the following well-known result of elementary geometry.

**Proposition 2.1.** *Two opposite angles in a chord quadrangle<sup>3</sup> add up to  $180^\circ$ .*

**Proof of Theorem 2.1.** Let  $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$  be an arrangement of four circles as described in Lemma 2.1. We show that it is not possible to add another circle  $\gamma_5$  such that:<sup>4</sup>

$$(1) \quad \begin{aligned} \text{cl}(\text{int } \gamma_1 \cap \text{int } \gamma_2) &\subseteq \text{int } \gamma_5, \\ \text{cl}(\text{int } \gamma_3 \cap \text{int } \gamma_4) &\subseteq \text{int } \gamma_5, \\ \text{cl}(\text{int } \gamma_2 \cap \text{int } \gamma_3) \cap \text{int } \gamma_5 &= \emptyset, \\ \text{cl}(\text{int } \gamma_4 \cap \text{int } \gamma_1) \cap \text{int } \gamma_5 &= \emptyset. \end{aligned}$$

Let the points  $S_i, S'_i$  ( $i = 1, 2, 3, 4$ ) and the angles  $\alpha, \beta, \gamma, \delta$  be defined as in Lemma 2.1. The conditions (1) imply

$$(2) \quad \begin{aligned} S_1, S'_1, S_3, S'_3 &\in \text{int } \gamma_5 \\ S_2, S'_2, S_4, S'_4 &\notin \text{int } \gamma_5. \end{aligned}$$

**Case 1:** *All angles in the quadrangle  $S'_1S_2S'_3S_4$  are  $< 180^\circ$ .*

Assume that there is a circle  $\gamma_5$  that contains  $S'_1, S'_3$  but not  $S_2, S_4$  in its interior. We show that  $\angle(S'_3S_4S'_1) + \angle(S'_1S_2S'_3) < \angle(S_4S'_1S_2) + \angle(S_2S'_3S_4)$ , which contradicts Lemma 2.1.

Let  $\tilde{S}'_1, \tilde{S}'_3$  be the intersection points of the straight line  $S'_1S'_3$  with  $\gamma_5$ , such that  $\tilde{S}'_1$  is nearer to  $S'_1$  than to  $S'_3$ . Note that due to the convexity of the quadrangle  $S'_1S_2S'_3S_4$  there are also two intersection points  $\tilde{S}_2, \tilde{S}_4$  of the straight line  $S_2S_4$  with  $\gamma_5$  (again, we assume that  $\tilde{S}_2$  is nearer to  $S_2$  than to  $S_4$ ). Evidently,  $\angle(\tilde{S}'_1S_2\tilde{S}'_3) > \angle(S'_1S_2S'_3)$  and  $\angle(\tilde{S}'_3S_4\tilde{S}'_1) > \angle(S'_3S_4S'_1)$ . Furthermore we have  $\angle(\tilde{S}'_1\tilde{S}_2\tilde{S}'_3) > \angle(\tilde{S}'_1S_2\tilde{S}'_3)$  and  $\angle(\tilde{S}'_3\tilde{S}_4\tilde{S}'_1) > \angle(\tilde{S}'_3S_4\tilde{S}'_1)$ . Applying Proposition 2.1 yields

$$\begin{aligned} 180^\circ &= \angle(\tilde{S}'_1\tilde{S}_2\tilde{S}'_3) + \angle(\tilde{S}'_3\tilde{S}_4\tilde{S}'_1) > \angle(\tilde{S}'_1S_2\tilde{S}'_3) + \angle(\tilde{S}'_3S_4\tilde{S}'_1) > \\ &> \angle(S'_1S_2S'_3) + \angle(S'_3S_4S'_1). \end{aligned}$$

Since the angles in the quadrangle  $S'_1S_2S'_3S_4$  add up to  $360^\circ$ , it follows that  $\angle(S_4S'_1S_2) + \angle(S_2S'_3S_4) > 180^\circ$ , which leads to the desired contradiction.

**Case 2:** *There is an angle  $\geq 180^\circ$  in the quadrangle  $S'_1S_2S'_3S_4$ .*

Since  $\beta$  and  $\delta$  are  $< 180^\circ$  we may assume without loss of generality that  $\alpha \geq 180^\circ$ . If  $\alpha = 180^\circ$ , then  $S_4$  lies in the convex hull of the points  $S'_1, S'_3$ , so that any circle with  $S'_1$  and  $S'_3$  in its interior also contains  $S_4$ , which violates (2). Thus let us assume that  $\alpha > 180^\circ$ . We distinguish two cases.

<sup>3</sup>A *chord quadrangle* is a quadrangle with all four vertices lying on a circle.

<sup>4</sup> $\text{cl } A$  denotes the closure of  $A$ ,  $\text{int } \gamma$  the interior of  $\gamma$ .

(a) *The angle  $\angle(S'_3S'_4S'_1)$  is  $< 180^\circ$ :*

In this case, the points  $S'_3, S'_4, S'_1, S_4$  form a quadrangle with all angles  $< 180^\circ$  (note that the straight line through  $S'_4S_4$  separates the points  $S'_3, S'_1$ ), so that its diagonals  $S'_3S'_1$  and  $S'_4S_4$  cut each other.

By convexity, it follows that any circle  $\gamma_5$  with  $S'_1, S'_3$  in its interior also contains a point  $\in \text{cl}(\text{int } \gamma_1 \cap \text{int } \gamma_4)$ , which violates the last condition of (1).

(b) *The angle  $\angle(S'_3S'_4S'_1)$  is  $\geq 180^\circ$ :*

We are going to show that the angles  $\angle(S'_1S'_4S'_3)$ ,  $\angle(S'_3S'_4S_3)$ ,  $\angle(S_3S'_4S_1)$ ,  $\angle(S_1S'_4S'_1)$  are all  $\leq 180^\circ$ . Then by Lemma 2.2, we may conclude that  $S'_4$  lies in the convex hull of the points  $S_1, S'_1, S_3, S'_3$ . It follows that any circle  $\gamma_5$  with  $S_1, S'_1, S_3, S'_3$  in its interior by convexity also contains  $S'_4$ , which contradicts (2).

Now let us take a look at the aforementioned angles:

◦  $\angle(S'_1S'_4S'_3)$ : This angle is by assumption  $\leq 180^\circ$ .

◦  $\angle(S_3S'_4S_1)$ : By Lemma 2.1,

$$\angle(S_3S'_4S_1) + \angle(S_1S'_2S_3) < 180^\circ < \angle(S'_4S_1S'_2) + \angle(S'_2S_3S'_4),$$

so that  $\angle(S_3S'_4S_1) < 180^\circ$ .

◦  $\angle(S'_3S'_4S_3)$ : Since  $S'_4$  is not contained in any circle, when walking counterclockwise on  $\gamma_4$  starting in  $S'_4$ , we first pass  $S'_3$  and then  $S_3$ . Hence,  $\angle(S'_3S'_4S_3)$  is the angle of a triangle inscribed in  $\gamma_4$ . It follows that  $\angle(S'_3S'_4S_3) < 180^\circ$ .

◦  $\angle(S_1S'_4S'_1)$ : An analogous argument as in the previous case shows that  $\angle(S_1S'_4S'_1) < 180^\circ$ .  $\square$

## REFERENCES

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