# A Note on Convex Realizability of Arrangements of Pseudocircles


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**Abstract**

An arrangement of pseudocircles is a collection of Jordan curves in the plane with at most two intersections between any two curves. We consider the question when such an arrangement can be realized with convex curves. We show that the existence of an open region which is contained in the interior of all curves of the arrangement is a sufficient condition for the existence of a convex realization.


## 1 Introduction

An *arrangement of pseudocircles* is a finite set $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of Jordan curves in the plane such that

(i) no three curves meet each other at the same point,

(ii) each two curves $\gamma_i, \gamma_j$ have at most two points in common, and

(iii) these *intersection points* in $\gamma_i \cap \gamma_j$ are always points where $\gamma_i, \gamma_j$ cross each other.

Any arrangement induces a planar embedding of a graph whose vertices are the intersection points and whose edges are the curve segments between these intersections. In the following we will often refer to this graph when talking about *vertices*, *edges*, and *faces* of the arrangement. Further, this graph establishes a cell complex. We say that two arrangements are *isomorphic* if they have the same associated cell complex.

**Definition 1.1.** Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be an arrangement of pseudocircles. The *weight of a face $F$* is the number of pseudocircles $\gamma_i$ such that $F$ is contained in $\mathrm{int}(\gamma_i)$, the interior of $\gamma_i$. The number of faces of given weight $k$ ($0 \leq k \leq n$) will be denoted by $f_k$. The vector $(f_0, f_1, \ldots, f_n)$ is called the *weight vector* of the arrangement.

It has already been shown that there are arrangements of pseudocircles that cannot be realized with proper circles [3], i.e. that are not isomorphic to any arrangement of proper circles. It is natural to go one step further and ask whether each arrangement can be realized with convex pseudocircles. Bultena et al. [2] have already shown that this holds for a special class of *intersecting families* of Jordan curves. Here an intersecting family of Jordan curves is a collection of Jordan curves such that there is at least one open region common to the interiors of all the curves and the above conditions (i) and (iii) hold, while (ii) is replaced with the condition that each two curves have a *finite* number of intersection points.

**Theorem 1.2** (Bultena et al. [2])**.** *Each intersecting family of $n$ Jordan curves with $f_0 = f_n = 1$ can be realized with convex Jordan curves.*

We use this theorem to show the following result.

**Theorem 1.3.** *Let $\Gamma$ be an arrangement of $n$ pseudocircles. If $f_n > 0$, then $\Gamma$ is realizable with convex pseudocircles.*

## 2 Proof of Theorem 1.3

We prove the theorem by first showing that there is at most one face of weight of weight $n$ (Proposition 2.1).[1] The proof is completed by

[1]As we unfortunately only found out shortly after the paper was published Proposition 2.1 is already known. It is an immediate consequence of the follow-

establishing that any arrangement with a face of weight $n$ has only a single face of weight 0 (Corollary 2.3).

**Proposition 2.1.** *For all arrangements of $n$ pseudocircles in the plane, $f_n \leq 1$.*

*Proof.* The case $n = 2$ is easy. If the two pseudocircles do not intersect, it is obvious that $f_2$ is either 0 or 1. On the other hand, there is only one arrangement of two intersecting pseudocircles (up to isomorphy), which has a single face of weight 2.

Given three curves $\gamma_1$, $\gamma_2$, $\gamma_3$, let us assume that they form more than one face of weight 3. This can only be the case if $\gamma_3$ cuts out at least two portions from the region $R := \mathrm{int}(\gamma_1) \cap \mathrm{int}(\gamma_2)$. Thus $\gamma_3$ must have at least four intersection points with $\partial R$. Since $\gamma_3$ cannot have more than two intersections with each $\gamma_i$ ($i = 1, 2$), there are exactly four intersections of $\gamma_k$ with $\partial R$, two of which are placed on each $\gamma_i$ ($i = 1, 2$). There are exactly two ways these vertices can be connected with edges of $\gamma_3$ shown in Figure 1.

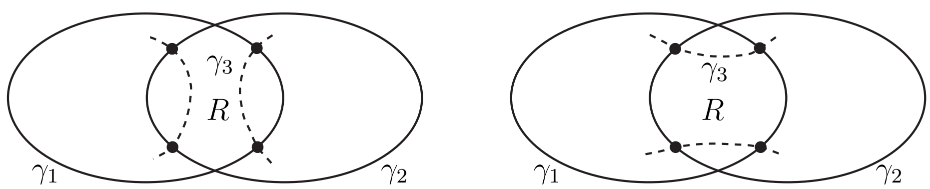


Figure 1: Connecting the vertices of $\gamma_3$.

In the first case (left picture) it is obvious that $\gamma_3$ must have additional intersections with $\gamma_1$ and $\gamma_2$, so that the pseudocircles

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ing theorem of J. Molnár, an improvement of the topological version of Helly's theorem in the plane.

**Theorem** (Molnár [4], see e.g. also [1] for a statement).
*Let $\mathcal{C}$ be a family of simply connected compact sets in the plane such that every two members of $\mathcal{C}$ have a connected intersection and every three members of $\mathcal{C}$ have nonempty intersection. Then $\cap\{C | C \in \mathcal{C}\}$ is nonempty and simply connected.*

In our terminology this implies that for any arrangement of pseudocircles $\Gamma$ in which each three pseudocircles have a common interior, $f_n(\Gamma) = 1$ (where $n = |\Gamma|$). Trivially, if $f_n > 0$ then each three pseudocircles have a common interior, so that generally $f_n \leq 1$, which is the statement of Proposition 2.1.

have more vertices than allowed. A similar contradiction occurs in the other case (right picture), where we can only have two faces of weight 3, if $\gamma_3$ cuts either $\gamma_1$ or $\gamma_2$ in at least four points.

Proceeding by induction, we assume that the proposition is true for $\leq n$ curves. Now suppose we have found a counterexample $\Gamma$ for the case $n+1$, i.e. $\Gamma = \{\gamma_1, \dots, \gamma_{n+1}\}$ with $f_{n+1} \geq 2$. Thus, $\bigcap_{i=1}^{n+1} \mathrm{int}(\gamma_i)$ consists of at least two faces. By inductive assumption, $K := \bigcap_{i=1}^{n-1} \mathrm{int}(\gamma_i)$ consists only of one face, the single face of weight $(n-1)$ of the arrangement $\Gamma' := \Gamma \backslash \{\gamma_n, \gamma_{n+1}\}$. Similarly, $K \cap \mathrm{int}(\gamma_n)$ and $K \cap \mathrm{int}(\gamma_{n+1})$ are the single faces of weight $n$ of the arrangements $\Gamma \backslash \{\gamma_{n+1}\}$ and $\Gamma \backslash \{\gamma_n\}$, respectively. Finally, we have already seen that $\mathrm{int}(\gamma_n) \cap \mathrm{int}(\gamma_{n+1})$ is the single face of weight 2 of the arrangement $\{\gamma_n, \gamma_{n+1}\}$. We claim that our assumption that $\bigcap_{i=1}^{n+1} \mathrm{int}(\gamma_i) = K \cap \mathrm{int}(\gamma_n) \cap \mathrm{int}(\gamma_{n+1})$ consists of at least two faces leads to a similar contradiction as the case $n = 3$ above. Essentially, we can apply the argumentation for the case $n = 3$ replacing $\gamma_3$ with $\partial K$ and $\gamma_1, \gamma_2$ with $\gamma_n, \gamma_{n+1}$. The important exception is that $\partial K$ may have more than two intersection points with $\gamma_n$ and $\gamma_{n+1}$. In this case, consider the arrangement of $\partial K$ and the boundary of $R' = \mathrm{int}(\gamma_n) \cap \mathrm{int}(\gamma_{n+1})$ (see Figure 2).

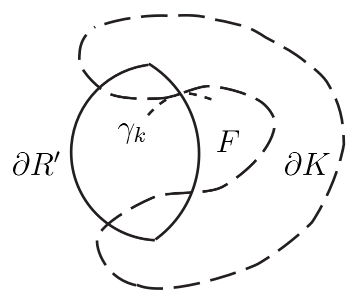


Figure 2: $\partial K$ and the boundary of $R' = \mathrm{int}(\gamma_n) \cap \mathrm{int}(\gamma_{n+1})$.

As should be clear from the figure, there is a face $F$ of weight 0 enclosed by these two curves. Let $V$ be one of the vertices on the boundary of this face. $V$ is the intersection of either $\gamma_n$ or $\gamma_{n+1}$ with another curve $\gamma_k$. Since $K$ is contained in the interior of all curves $\gamma_i$ $(i < n)$, $K \subset \mathrm{int}(\gamma_k)$. Thus, the course of $\gamma_k$ must be as follows (cf. Figure 2): After entering $F$ the curve must leave $F$ again, causing a second vertex with $\partial R'$. Afterwards $\gamma_k$ must surround $K$, which can only be done by creating two further intersections with $\partial R'$. Therefore $\gamma_k$ has at least four vertices on $\partial R'$, which leads (as shown in the case $n = 3$) to a contradiction. $\square$

**Proposition 2.2** (Yang [5])**.** *Let $\Gamma$ be an arrangement of $n$ pseudocircles in the plane with weight vector $(f_0, f_1, \dots, f_n)$ and $f_n > 0$. Then there is an arrangement of pseudocircles $\Gamma'$ with weight vector $(f'_0, f'_1, \dots, f'_n) = (f_n, f_{n-1}, \dots, f_0)$.*

*Proof.* Choose an arbitrary point $P$ contained in the face of weight $n$. We place a sphere on the plane touching it in $P$. By stereographic projection from $P^*$, the antipodal point of $P$, we obtain an arrangement $\Gamma_{S^2}$ of pseudocircles on $S^2$. Obviously, when projecting an arrangement from the plane to the sphere, we want the interiors of pseudocircles in the plane to be projected to the interiors of pseudocircles on the sphere. Thus, the arrangement on the sphere has the same weight vector as $\Gamma$. Now consider the arrangement $\Gamma'_{S^2}$ arising when we swap interior and exterior of each pseudocircle. Then an arbitrary point $Q$ on $S^2$ is contained in the interior of $\gamma$ in $\Gamma'_{S^2}$ if and only if $Q$ is outside $\gamma$ in $\Gamma_{S^2}$. Therefore, a face of weight $k$ in $\Gamma'_{S^2}$ has weight $n - k$ in $\Gamma_{S^2}$ and vice versa. In particular, $P$ is not contained in the interior of any pseudocircle in $\Gamma'_{S^2}$. Thus, we can project the whole arrangement from $P$ to the plane tangent to $P^*$ yielding an arrangement $\Gamma'$ with the claimed property. $\square$

The following result is an immediate consequence of Propositions 2.1 and 2.2. It finishes the proof of Theorem 1.3.

**Corollary 2.3.** *In any arrangement of $n$ pseudocircles with $f_n > 0$, it holds that $f_0 \leq 1$.*

*Proof.* Any arrangement with weight vector $(f_0, f_1, \dots, f_n)$ where $f_0 > 1$ and $f_n > 0$ would by Proposition 2.2 imply the existence of an arrangement with $f_n > 1$, which contradicts Proposition 2.1. $\square$

# 3 Open Problems

Theorem 1.3 proves convex realizability only for a special instance of Theorem 1.2. The main question posed in [2] is still open. For which arrangements of Jordan curves are there convex realizations? In particular, concerning arrangements of pseudocircles it is still open whether all of them can be realized with convex pseudocircles or not.

# References


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[4] J. Molnár, Über den zweidimensionalen topologischen Satz von Helly (in Hungarian, Russian and German summary), *Mat. Lapok* 8 (1957), 108–114.

[5] Y. Yang, Arrangements of Circles on $E^2$, unpublished typescript.