Improved Upper Bounds on the Number of Vertices of Weight $\leq k$ in Particular Arrangements of Pseudocircles

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Abstract

In arrangements of pseudocircles (Jordan curves) the weight of a vertex (intersection point) is the number of pseudocircles that contain the vertex in its interior. We give improved upper bounds on the number of vertices of weight $\leq k$ in certain arrangements of pseudocircles in the plane.

1 Introduction

A *pseudocircle* is a simple closed (Jordan) curve in the plane. An *arrangement of pseudocircles* is a finite set $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of simple closed curves in the plane such that (i) no three curves meet each other at the same point, (ii) each two curves γ_i, γ_j have at most two points in common, and (iii) these *intersection points* in $\gamma_i \cap \gamma_j$ are always points where γ_i, γ_j cross each other. An arrangement is *complete* if each two pseudocircles intersect.

Any arrangement can be interpreted as a planar embedding of a graph whose vertices are the intersection points between the pseudocircles and whose edges are the curves between these intersections. In the following we will often refer to this graph when talking about *vertices*, *edges*, and *faces* of the arrangement.

Definition 1 *Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be an arrangement of pseudocircles. The* weight of a vertex *V is the number of pseudocircles γ_i such that V is contained in $\mathrm{int}(\gamma_i)$, the interior of γ_i. Weights of edges and faces are defined accordingly.*

We will consider the number $v_k = v_k(\Gamma)$ of vertices of given weight k, the number $v_{\leq k} = v_{\leq k}(\Gamma)$ of vertices of weight $\leq k$, and the number $v_{\geq k} = v_{\geq k}(\Gamma)$ of vertices of weight $\geq k$. Further, $f_k = f_k(\Gamma)$ denotes the number of faces of weight k.

Concerning the characterization of the *weight vectors* $(v_0, v_1, \ldots, v_{n-2})$ of arrangements of pseudocircles little is known. So far, sharp upper bounds on v_k exist only for $k = 0$.

Theorem 1 (Kedem et al. [2]) *For all arrangements Γ with $n := |\Gamma| \geq 3$,*

$$v_0 \;\leq\; 6n - 12.$$

Moreover, for each $n \geq 3$ there is an arrangement of n (proper) circles in the plane such that $v_0 = 6n - 12$.

Theorem 1 can be used to obtain general upper bounds on $v_{\leq k}$ by some clever probabilistic methods.

Theorem 2 (Sharir [5]) *For all arrangements of n pseudocircles and all $k > 0$,*

$$v_{\leq k} \;\leq\; 26kn.$$

On the other hand, J. Linhart and Y. Yang established the following sharp upper bound on $v_{\geq k}$.

Theorem 3 (Linhart, Yang [4]) *For all arrangements of $n \geq 2$ pseudocircles and all k with $0 \leq k \leq n-2$,*

$$v_{\geq k} \;\leq\; (n+k)(n-k-1).$$

In this paper we are going to improve the upper bounds of Theorems 1 and 2 for some particular classes of arrangements.

2 Preparations and First Results

2.1 Improved Bounds from Theorem 3

We start with a bound due to J. Linhart, which holds if there is a face of large weight in the arrangement. It is based upon the following result of Y. Yang (which can be shown by turning the arrangement in question inside out).

Proposition 4 (Yang [6]) *Let Γ be an arrangement of n pseudocircles in the plane with weight vector $(v_0, v_1, \ldots, v_{n-2})$ and $f_n > 0$. Then there is an arrangement of pseudocircles Γ' with weight vector $(v'_0, v'_1, \ldots, v'_{n-2}) = (v_{n-2}, v_{n-1}, \ldots, v_0)$.*

J. Linhart [3] pointed out that Proposition 4 together with Theorem 3 yields the following improvement of the upper bound on $v_{\leq k}$ for arrangements with $f_n > 0$.

Theorem 5 (Linhart [3]) *For all arrangements Γ of n pseudocircles with $f_n > 0$,*

$$v_{\leq k} \;\leq\; 2(k+1)n - (k+1)(k+2).$$

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Proof. Let Γ be an arrangement with weight vector $(v_0, v_1, \ldots, v_{n-2})$ and $f_n > 0$. Then by Proposition 4, there exists an arrangement Γ' with $v'_k = v_{n-k-2}$ vertices of weight k for $0 \leq k \leq n-2$. Therefore, by Theorem 3,

$$\begin{aligned} v_{\leq k} &= \sum_{j=0}^{k} v_j = \sum_{j=0}^{k} v'_{n-j-2} = \sum_{j=n-k-2}^{n-2} v'_j \\ &= v'_{\geq n-k-2} \\ &\leq (n+n-k-2)\big(n-(n-k-2)-1\big) = \\ &= 2(k+1)n-(k+1)(k+2). \end{aligned}$$

□

Theorem 5 may be used to obtain bounds of

$$v_{\leq k} \leq 2n(n-w+k+1)-(k+1)(k+2),$$

if there is a face of large weight w. Of course, in general this bound is worse than that of Theorem 2 as it is quadratic in n.

2.2 Improved Bounds from Theorem 1

2.2.1 Faces with many Participating Pseudocircles

Bounds on v_0 can also be improved if there is a face of weight 0 with many pseudocircles participating in its boundary.

Proposition 6 *Let Γ be an arrangement of n pseudocircles with a face F of weight 0 such that for each $\gamma \in \Gamma$ there is an edge of γ on ∂F, the boundary of F. Then*

$$v_0 \leq 4n-6.$$

Proof. Let us assume that there exists an arrangement Γ as described in the proposition such that $v_0 > 4n-6$. As shown in Fig. 1 we can add a pseudocircle γ' to Γ such that γ' cuts each $\gamma \in \Gamma$ on ∂F in two vertices of weight 0. Note that we may add γ' such that it does not contain any vertices of Γ in its interior. Hence, in the arrangement $\Gamma' := \Gamma \cup \{\gamma'\}$ we have

$$v_0(\Gamma') > 4n-6+2n = 6(n+1)-12,$$

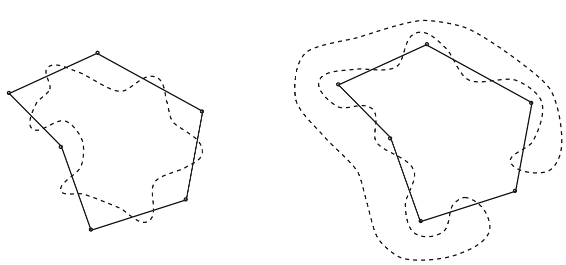

Figure 1: Adding a pseudocircle γ' cutting each $\gamma \in \Gamma$ in two vertices of weight 0.

which contradicts Theorem 1. □

This proof method can be generalized to obtain a bound of $v_0 \leq 6n-2m-6$ for arrangements of n pseudocircles with a face of weight 0 in whose boundary m pseudocircles participate.

As shown in Fig. 2, the bound of Proposition 6 is sharp.

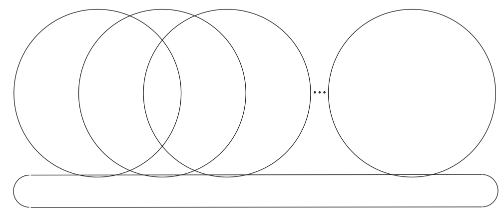

Figure 2: Arrangement of n pseudocircles with $v_0 = 4n-6$. Note that each pseudocircle participates in the unbounded face of weight 0.

2.2.2 A Bound Depending on f_0

Theorem 1 can also be used to obtain an upper bound on v_0 that depends on the number f_0 of faces of weight 0.

Theorem 7 *Let Γ be an arrangement of n pseudocircles. Then*

$$v_0 \leq 2n+2f_0-4.$$

Theorem 7 can be proved from Theorem 1 with the aid of the upper bound on f_0 given in Proposition 8 below, which is also a direct consequence of Theorem 1. As Theorem 7 together with Proposition 8 entails Theorem 1, this can be considered as self-strengthening of Theorem 1.

Proposition 8 *Let Γ be an arrangement of $n \geq 3$ pseudocircles. Then*

$$f_0 \leq 2n-4.$$

Proof. First note that the boundary of each face of weight 0 consists of at least three edges (and hence vertices) of weight 0. For if there were a face with only two edges belonging to some pseudocircles γ_i and γ_j, then $\gamma_i \cap \gamma_j$ would have more than the two allowed intersection points. On the other hand, each vertex of weight 0 is on the boundary of only a single face of weight 0. Therefore by Theorem 1,

$$f_0 \leq \frac{v_0}{3} \leq \frac{6n-12}{3} = 2n-4.$$

□

Proof of Theorem 7. If all faces of weight 0 are triangles, then $v_0 = 3f_0$, so that by Proposition 8,

$$v_0 = 2f_0 + f_0 \leq 2f_0 + 2n - 4.$$

We proceed by induction on the number of faces of weight 0 with $\ell > 3$ edges. Adding a pseudocircle γ as described in the proof of Proposition 6 removes such a face, while adding $\ell - 1$ new faces of weight 0 and 2ℓ new vertices of weight 0. Then by induction assumption

$$v_0 + 2\ell \ \leq\ 2(n+1) + 2(f_0 + (\ell - 1)) - 4,$$

whence the theorem follows. □

3 Improved Bounds for Complete Arrangements with Forbidden Subarrangements

In this section we consider bounds for complete arrangements. First, we'd like to remark that the bound of Theorem 1 is sharp for complete arrangements, too. That is, for each $n \geq 3$ there is a complete arrangement with $v_0 = 6n - 12$ (see Fig. 3).

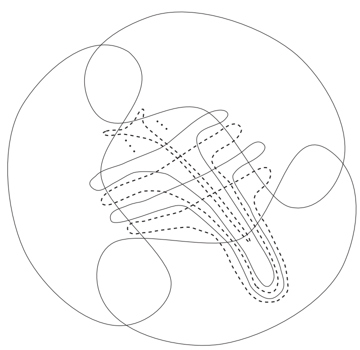

Figure 3: A complete arrangement of pseudocircles with $v_0 = 6n - 12$.

Thus, in order to obtain improved bounds on v_0, one has to put some additional restrictions on the arrangement, e.g. by forbidding certain subarrangements, which will be considered in the following.

3.1 Forbidding α-subarrangements

Evidently, arrangements of three pseudocircles are the smallest subarrangements of interest in this respect. Figure 4 shows the four different types one has to take into account.

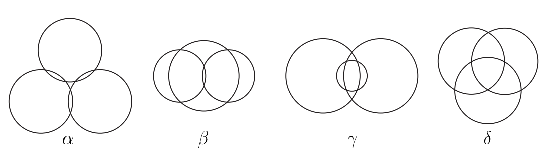

Figure 4: Complete arrangements of three pseudocircles in the plane.

Subarrangements of type α play a special role here. Not only are they the only arrangements of three pseudocircles which meet the bound of Theorem 1. They are also the only complete arrangements of three pseudocircles without any face of weight 3, which is of importance in the light of the following Helly type theorem.

Theorem 9 (Helly [1]; Kerékjártó) *Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be an arrangement of pseudocircles such that for all pairwise distinct $\gamma_i, \gamma_j, \gamma_k$,*

$$\operatorname{int}(\gamma_i) \cap \operatorname{int}(\gamma_j) \cap \operatorname{int}(\gamma_k) \neq \varnothing.$$

Then

$$\bigcap_{i=1}^{n} \operatorname{int}(\gamma_i) \neq \varnothing.$$

Corollary 10 *Let Γ be a complete arrangement of $n \geq 2$ pseudocircles that has no subarrangement of type α. Then*

$$v_{\leq k} \ \leq\ 2(k+1)n - (k+1)(k+2).$$

Proof. Since Γ has no α-subarrangement, the condition in Theorem 9 holds, and we may conclude that there is a face of weight n in Γ. Applying Theorem 5 yields the claimed bound. □

3.2 Forbidding α^4-subarrangements

It is a natural question whether there are alternative bounds for other forbidden subarrangements as well. The unique complete arrangement of four pseudocircles that meets the bound of Theorem 1 seems to be a good candidate. In such an α^4-*arrangement* each subarrangement of three pseudocircles is of type α. α^4-arrangements prominently appear in the arrangement of Fig. 3, where the three outer pseudocircles together with any other pseudocircle form an α^4-arrangement. Indeed, for α^4-free arrangements in which there is also no β-subarrangement (cf. Fig. 4) we can show the following improved upper bound on v_0.

Theorem 11 *In complete arrangements of $n \geq 2$ pseudocircles that are α^4-free and β-free,*

$$v_0 \ \leq\ 4n - 6.$$

Theorem 11 follows immediately from the following bound on f_0 together with Theorem 7.

Theorem 12 *In complete arrangements of $n \geq 2$ pseudocircles that are α^4-free and β-free,*

$$f_0 \ \leq\ n - 1.$$

For the proof of Theorem 12 the following lemma is useful. We skip a proof.

Lemma 13 *Let Γ be a complete, β-free arrangement. Then for each face F of weight 0 in Γ there is a unique α-arrangement $\Gamma_\alpha \subseteq \Gamma$ such that F is the bounded face of weight 0 in Γ_α. In particular, each face of weight 0 has only three edges.*

Proof of Theorem 12. We give a proof by induction on $n := |\Gamma|$. The case $n = 2$ is trivial, while for $n = 3$ one may consult Fig. 4. If $n > 3$, choose an arbitrary pseudocircle γ in Γ. By induction assumption the theorem holds for $\Gamma' := \Gamma \setminus \{\gamma\}$. We claim that adding γ to Γ' will increase f_0 by at most 1. Indeed, f_0 could be increased by more than 1 only in one of the following two cases:

First, γ may separate a single face F of weight 0 in Γ' into more than two new faces of weight 0. By Lemma 13 such a face F has only three edges which belong to three pseudocircles that form an α-arrangement Γ_α. Thus, in order to separate F as described above, γ has to intersect each pseudocircle of Γ_α in two vertices of weight 0, so that $\Gamma_\alpha \cup \{\gamma\}$ would be a forbidden α^4-arrangement.

On the other hand, there might be two distinct faces F_1, F_2 in Γ', such that γ separates each F_i into two new faces of weight 0. By Lemma 13, there is a unique α-arrangement Γ_α enclosing F_1, so that F_2 will be outside Γ_α (i.e. contained in the unbounded face of weight 0 of Γ_α). Hence, γ would have to intersect the bounded as well as the unbounded face of weight 0 in Γ_α. But it is easy to see that this can only happen if γ together with two pseudocircles in Γ_α forms a forbidden β-subarrangement. □

The bounds of Theorems 11 and 12 are sharp. Take $(n-1)$ pseudocircles such that any subarrangement of three pseudocircles is of type δ. In this arrangement $f_0 = 1$, and each pseudocircle has an edge (and hence two vertices) on the single face of weight 0. Adding another pseudocircle just as indicated in the proof of Proposition 6 (cf. Fig. 1) yields an arrangement with $f_0 = n - 1$ and $v_0 = 4n - 6$.

The improved upper bound on v_0 of Theorem 11 can in turn be used to improve the upper bound on $v_{\leq k}$ for complete, α^4-free arrangements.

Theorem 14 *For complete, α^4-free and β-free arrangements of $n \geq 2$ pseudocircles and $k > 0$,*

$$v_{\leq k} \;\leq\; 18kn.$$

Proof. The proof is basically identical to the proof of Theorem 2 in [5], only with the application of Theorem 1 replaced by an application of Theorem 11 and the constants adapted accordingly. □

The bounds of Theorems 11 and 14 can easily be generalized to (not necessarily complete) β-free arrangements that do not contain certain subarrangements that are generalizations of α^4-arrangements.

4 Conclusion

We conjecture that Theorems 11, 12, and 14 also hold if we drop the condition that the arrangement is β-free, i.e., for the improved bounds to hold it is sufficient that a complete arrangement is α^4-free. However, the topology of these arrangements quickly becomes rather involved so that we haven't yet succeeded in proving this. As an α^4-arrangement cannot be realized with unit circles, a proof of our conjecture would also imply that Theorems 11 and 14 hold in particular for complete arrangements of unit circles. As shown in Fig. 5, in this case the improved bound on v_0 would also be sharp.

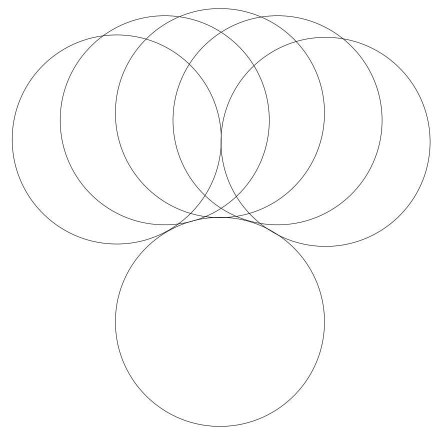

Figure 5: Complete arrangement of six unit circles with $v_0 = 4n - 6$. Points that look like touching points should be two intersection points between the respective circles. Further circles can easily be added to meet the bound for arbitrary n.

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